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Quotient compact images of metric spaces, and related matters

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Abstract

As a generalization of developments of (developable) spaces, we introduce the notion of σ-strong networks of spaces, and related notions. We give new characterizations for around quotient compact images of metric spaces by means of σ-strong networks, and give some related matters. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and definitions

Some characterizations for certain quotient s-images of metric spaces are obtained by means of cs^*-networks or cs-networks. For example, see [18,27]; and [19] for quotient s-images; and sequence-covering, quotient s-images respectively. On the other hand, characterizations for the quotient compact images of metric spaces are also obtained by means of weak bases. See [17,29], for example. We assume that spaces are Hausdorff, and maps are continuous surjections.

Let us introduce the following notion of σ-strong networks as a generalization of "development" in developable spaces, and consider certain quotient image of metric spaces in terms of σ-strong networks.

Definition. Let $\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ be a sequence of covers of a space X such that $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$ (simply, $\mathcal{C}_{n+1} \prec \mathcal{C}_n$) for each $n \in \mathbb{N}$. Let us call $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ a *σ-strong network* for

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X, if $\{St(x, \mathcal{C}_n) \mid n \in \mathbb{N}\}$ is a local network at x in X; that is, for any $x \in X$, and for any $C_n \in \mathcal{C}_n$ with $C_n \ni x$, $\{C_n \mid n \in \mathbb{N}\}$ is a local network at x in X. If the $\mathcal{C}_n$ are open covers, then such a sequence $\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is called a development for X.

In this paper, by means of σ-strong networks, we give new characterizations for the quotient and the following types of images of metric spaces, and give some related matters around these images of metric spaces:

compact images; sequence-covering compact images; π-images; *sequence-covering* π-*images*; s- *and* π-*images, etc*.

For example, we have the following statements, which are proved in Theorems 4, 9, and 12, respectively.

(A) X is a quotient compact image of a metric space $\Longleftrightarrow$ X is a sequential space with a σ-point-finite strong cs^*-network $\Longleftrightarrow$ X is a sequential space with a σ-point-finite strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where each $\mathcal{C}_n$ determines X.

(B) X is a sequence-covering, quotient compact image of a metric space $\Longleftrightarrow$ X is a sequential space with a σ-point-finite strong cs-network $\Longleftrightarrow$ X has a point-regular weak base.

(C) X is a quotient π-image of a metric space $\Longleftrightarrow$ X is a sequential space with a σ-strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where each $\mathcal{C}_n$ determines X.

Definition. Let $f : X \to Y$ be a map. Then,

(i) f is called a *compact map* (respectively *s-map*) if any $f^{-1}(y)$ is compact (respectively separable). If X is a metric space with a metric d, then f is called a π-*map* (= P-*map*) (with respect to d) if, for any $y \in Y$, and for any open nbd U of y, $d(f^{-1}(y),\ X - f^{-1}(U)) > 0$; equivalently, if $d(x_n, a_n) \to 0$ with $f(a_n) = y$, then $f(x_n) \to y$. Obviously, every compact map from a metric space is a π-map.

(ii) f is *compact-covering* if every compact subset of Y is the image of some compact subset of X. For compact-covering maps, see [22,23], etc. Also, f is called *sequence-covering* [25], if for each convergent sequence L (including its limit point) in Y, there exists a convergent sequence K in X with $f(K) = L$. Obviously, every open map from a first-countable space is sequence-covering. Not every compact-covering map is sequence-covering, and also, not every sequence-covering map is compact-covering [25]. While, in [11], a map $f : X \to Y$ is called sequence-covering, if for each convergent sequence L in Y, there exists a compact subset (not necessarily a convergent sequence) K of X such that $f(K) = L$. Let us call such a sequence-covering map of [11] *pseudo-sequence-covering* in this paper. If the $f(K)$ is a subsequence of L, then such a map f is called *subsequence-covering* [20]. Compact-covering maps, sequence-covering maps, and related maps have played important roles in the theory of k-networks, cs^*-networks, cs-networks; and (weak) Cauchy sequences, etc, among certain quotient images of metric spaces. Clearly, for a map $f : X \to Y$, f is compact-covering, or sequence-covering $\Rightarrow$ f is pseudo-sequence-covering $\Rightarrow$ f is subsequence-covering.

Definition. Let X be a space X. For a cover $\mathcal{C}$ of X, X *is determined by* $\mathcal{C}$ [11], if $G \subset X$ is open in X iff $G \cap C$ is open in C for every $C \in \mathcal{C}$. We use "X is determined by $\mathcal{C}$" instead of the usual "X has the weak topology with respect to $\mathcal{C}$". Every space X is determined by any open cover of X. Recall that a space X is *sequential* iff X is determined by a cover of compact metric subsets. It is well known that a sequential space is characterized as the quotient image of a (locally compact) metric space [9].

Definition. A cover $\mathcal{C}$ of a space X is called a *cs^*-network* (respectively *cs-network*) if, for any open nbd G of x, and for any sequence L converging to x, there exists $C \in \mathcal{C}$ such that $x \in C \subset G$, and C contains L frequently (= cofinally) (respectively eventually (= residually)). For $x \in X$, let us call such a cover $\mathcal{C}$ a *cs^*-network* (respectively *cs-network*) *at* x. Any cs-network is a cs^*-network. As for spaces with point-countable cs^*- or cs-networks, see [19,21,27], etc.

Definition. Let X be a space. Let $\mathcal{T} = \bigcup\{\mathcal{T}_x \mid x \in X\}$ be a family of collections of subsets in X satisfying the following: any $A \in \mathcal{T}_x$ contains x; for any $A, B \in \mathcal{T}_x$, there exists $C \in \mathcal{T}_x$ with $C \subset A \cap B$; and, $G \subset X$ is open in X iff for each $x \in G$, there exists $T \in \mathcal{T}_x$ such that $T \subset G$. Then, $\mathcal{T}$ is called a *weak base* for X, $\mathcal{T}_x$ a *weak nbd base* at x, and each element of $\mathcal{T}_x$ is called a *weak nbd* of x. If each $\mathcal{T}_x$ is countable, then such a space X is called *g-first countable.* Any open base is a weak base. Note that any weak base is a cs-network by Fact 1 below.

Definition. Let $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ be a σ-strong network for a space X. Let us say that $\mathcal{C}$ is a *σ-strong cs^*-network* (respectively *σ-strong cs-network*; *σ-strong weak base*) if the cover $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ of X is a cs^*-network (respectively cs-network; weak base). Note that any space X has a σ-strong cs-network; indeed, consider $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where $\mathcal{C}_1$ a base for X, and $\mathcal{C}_n = \{\{x\} \mid x \in X\}$ for $n \geqslant 2$.

We conclude this section by recording some elementary facts which will be used later on. These facts are routinely shown. (For Fact 1, if the conclusion does not hold, then there exist a weak nbd T of x, and a subsequence S of L with $S \cap T = \emptyset$, thus, $X - S$ is open in X, a contradiction. For Fact 4, the "only if" part is easy, and the "if" part is shown in [20] (without X being metric or sequential).)

Associated with Fact 2, let us call the map f in (1) (respectively (2)) the map *defined by* the cover $\mathcal{C}$ (respectively σ-strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$).

Fact 1. *Let L be a sequence converging to $x \notin L$. Then, any weak nbd T of x contains L eventually.*

Fact 2.

(1) *Let $\mathcal{C} = \{C_\alpha \mid \alpha \in I\}$ be a cover of X. Let S be the topological sum $\bigcup\{C_\alpha \times \{\alpha\} \mid \alpha \in I\}$. Let $f : S \to X$ define by $f((x, \alpha)) = x$. Then, f is a map such that $f^{-1}(x) = \{(x, \alpha) \mid x \in C_\alpha\}$. When $\mathcal{C}$ determines X, f is quotient.*

(2) *Let $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ be a σ-strong network for X. For each $n \in N$, let $\mathcal{C}_n = \{C_\alpha \mid \alpha \in I_n\}$, and let I_n be a discrete space. Let $M = \{(\sigma(n)) \in \Pi I_n \mid \{C_{\sigma(n)} \mid n \in \mathbb{N}\}$ is a*

network of some point x_σ in X}. Let $f : M \to X$ define by $f(\sigma) = x_\sigma$. Then M is a (0-dimensional) metric subspace of a Baire space $\Pi\{I_n \mid n \in \mathbb{N}\}$. Also, f is a map such that $f^{-1}(x) = \Pi\{A_n \mid n \in \mathbb{N}\}$, where $A_n = \{\alpha \in I_n \mid C_\alpha \ni x\}$.

Fact 3. *Let $f : X \to Y$ be a quotient map. If X is determined by $\mathcal{C}$, then Y is determined by $f(\mathcal{C}) = \{f(C) \mid C \in \mathcal{C}\}$.*

Fact 4. *Let $f : X \to Y$ be a map such that X is metric (or sequential). Then, f is quotient iff f is subsequence-covering, and Y is sequential.*

2. Results

First, for a cover $\mathcal{C}$ of a space X, let us consider the following conditions (c_1), (c_2), and (c_3). For gaps among these (c_i), see Remark 13(1).

(c_1) For each sequence $\{x_n \mid n \in \mathbb{N}\}$ converging to $x \in X$, some $C \in \mathcal{C}$ contains the point x, and a point x_n.
(c_2) Same as (c_1), but replace "a point x_n" by "points x_n frequently".
(c_3) Same as (c_1), but replace "a point x_n" by "points x_n eventually".

Lemma 1. *Let X be a sequential space. Let $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where each $\mathcal{C}_n$ is a cover of X with $\mathcal{C}_{n+1} \prec \mathcal{C}_n$. Consider the following properties:*
(a) *Any $\mathcal{C}_n$ determines X.*
(b) *Any $\mathcal{C}_n$ satisfies (c_2).*
(c) *Any $\mathcal{C}_n$ satisfies (c_1).*
(d) *For $G \subset X$, if for each $x \in G$, $St(x, \mathcal{C}_n) \subset G$ for some $n \in \mathbb{N}$, then G is open in X.*
(e) *For any $n \in \mathbb{N}$, $\bigcup\{\mathcal{C}_m \mid m \geqslant n\}$ is a cs^*-network for X.*
(f) *$\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is a cs^*-network for X.*

Then (1)–(4) *below hold.*
(1) (a) $\Longleftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d); *and* (e) $\Rightarrow$ (b) & (f) *hold.*
(2) *If $\mathcal{C}$ is a σ-strong network,* (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (e); *and* (c) $\Longleftrightarrow$ (d) *hold.*
(3) *If $\mathcal{C}$ is σ-point-finite,* (e) $\Longleftrightarrow$ (f) *holds (without X being sequential).*
(4) *If $\mathcal{C}$ is a σ-point-finite strong network (i.e., $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is a σ-strong network with each $\mathcal{C}_n$ point-finite),* (a)–(f) *are equivalent.*

Proof. (1) For (a) $\Rightarrow$ (b), let L be a sequence converging to $x \notin L$. Since L is not closed in X, by (a), for some $C \in \mathcal{C}_n$, $L \cap C$ is not closed in C. Thus, C contains x, and L frequently. Thus, (b) holds. For (b) $\Rightarrow$ (a), suppose that $G \subset X$ is not open in X. Since X is sequential, for some $x \in G$, some sequence S in $X - G$ converges to x. Then, by (b), some $C \in \mathcal{C}_n$ contains x, and S frequently. Thus, $C \cap G$ contains x, but does not meet S. Then, $C \cap G$ is not open in C. Thus, (a) holds. For (c) $\Rightarrow$ (d), suppose that the hypothesis of (d) holds, but the $G \subset X$ is not open in X, thus, for some $x \in G$, some sequence S in $X - G$ converges to x. While, the hypothesis of (d), for some $m \in \mathbb{N}$, $St(x, \mathcal{C}_m) \subset G$. But, by (c), some $C \in \mathcal{C}_m$

contains x, and a point $p \in S$. Then, $p \in G \cap S$, a contradiction. Thus, (d) holds. (b) ⇒ (c), and (e) ⇒ (f) are clear. (e) ⇒ (b) is obvious, for $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ for each $n \in \mathbb{N}$.

(2) For (b) ⇒ (e), let U be an open nbd of x, and L be a sequence converging to x. Let $n \in \mathbb{N}$. Then, for some $m \geqslant n$, $St(x, \mathcal{C}_m) \subset U$, for $\bigcup\{\mathcal{C}_n\colon n \in \mathbb{N}\}$ is a σ-strong network. But, by (b), some $C \in \mathcal{C}_m$ contains x, and L frequently. Thus, $x \in C \subset U$, and C contains L frequently. Thus, (e) holds. For (d) ⇒ (c), let L be a sequence converging to $x \notin L$. Since $X - L$ is not open in X, by (d), for some $y \in X - L$, any $St(y, \mathcal{C}_n)$ meets L. But, $y = x$, because $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is a σ-strong network, and $L \cup \{x\}$ is compact. Thus, any $St(x, \mathcal{C}_n)$ meets L, which shows that (c) holds.

(3) For (f) ⇒ (e), let U be an open nbd of x, and L be a sequence converging to $x \in L$. Let $n \in \mathbb{N}$. Then, by (f), for some $m \in \mathbb{N}$, and some $C \in \mathcal{C}_m$, $x \in C \subset U$, and C contains L frequently. If $m < n$, we put $\{C \in \bigcup\{\mathcal{C}_i \mid i < n\} \mid x \in C\} = \{C_j \mid j \leqslant k\}$ by the point-finiteness of these $\mathcal{C}_i$ $(i < n)$. For each $j \leqslant k$, if C_j contains a point $p_j \neq x$, then choose the point p_j, and let F be the finite set of these points p_j. Then, for some $\ell \in \mathbb{N}$, and some $D \in \mathcal{C}_\ell$, $x \in D \subset U - F$, and D contains L frequently, which implies $\ell \geqslant n$. Hence, (e) holds.

(4) By (1)–(3), it suffices to show that (d) ⇒ (b) holds. But, in the proof of (d) ⇒ (c) in (2), actually, any $St(x, \mathcal{C}_n)$ meets L frequently, for $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is a σ-strong network. Then, for any $n \in \mathbb{N}$, some $C \in \mathcal{C}_n$ contains x, and L frequently, for $\mathcal{C}_n$ is point-finite. Thus, (d) ⇒ (b) holds. □

Remark 2. In the previous lemma, the σ-strong networkness of the $\mathcal{C}$ is essential in (a) ⇒ (e); indeed, let $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where $\mathcal{C}_1$ is a base containing X, $\mathcal{C}_n = \{X\}$ for $n \geqslant 2$. Also, the point-finiteness of the covers $\mathcal{C}_n$ for the σ-strong network $\mathcal{C}$ is essential in (c) or (d) ⇒ (a), and (f) ⇒ (c) or (e); indeed, (c) ⇒ (a) does not hold in view of Remark 13(1). To see (f) ⇒ (c) or (e) does not hold, consider a non-discrete sequential space X, and a σ-strong cs-network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ for X, where $\mathcal{C}_1$ is a base (or the cover of all convergent sequences with its limit point) in X, and $\mathcal{C}_n = \{\{x\} \mid x \in X\}$ $(n \geqslant 2)$.

The following holds in view of Lemma 1 together with Fact 1.

Proposition 3. *For a sequence $\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ of point-finite covers of a space X with $\mathcal{C}_{n+1} \prec \mathcal{C}_n$, the following are equivalent.*

(a) *For each $x \in X$, $\{St(x, \mathcal{C}_n) \mid n \in \mathbb{N}\}$ is a weak nbd base at x.*
(b) *X is a sequential space such that, for each $x \in X$, $\{St(x, \mathcal{C}_n) \mid n \in \mathbb{N}\}$ is a cs^*-network (or cs-network) at x.*
(c) *X is a sequential space, and $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is a σ-strong cs^*-network.*
(d) *X is a sequential space, and $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is a σ-strong network, and any $\mathcal{C}_n$ determines X.*

Now, let us give new characterization for the quotient compact images of metric spaces.

Theorem 4. *The following are equivalent for a space X.*

(a) *X is a pseudo-sequence-covering, quotient compact image of a metric space.*
(b) *X is a quotient compact image of a metric space.*
(c) *X is a sequential space with a σ-point-finite strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ determines X.*
(d) *X is a sequential space with a σ-point-finite strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_1) (or (c_2)).*
(e) *X is a sequential space with a σ-point-finite strong cs^*-network.*

Proof. (a) $\Rightarrow$ (b) is clear. (b) $\Rightarrow$ (c) holds by means of Fact 3. And, (c) $\Longleftrightarrow$ (d) $\Longleftrightarrow$ (e) holds by Lemma 1. For (d) $\Rightarrow$ (a), let $f : M \to X$ be the map defined by the σ-point-finite strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ (see Fact 2(2)). We show the map f is pseudo-sequence-covering. Let $L = \{x_n \mid n \in \mathbb{N}\}$ be a sequence converging to $x \notin L$. Let $\mathcal{C}_n = \{C_\alpha \mid \alpha \in A_n\}$, and let $B_n = \{\alpha \in A_n \mid C_\alpha \ni x\}$ for each $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, define $\sigma_m = (\sigma_m(n)) \in M$ as follows: if $St(x, \mathcal{C}_n) \ni x_m$, pick $\alpha \in B_n$ with $C_\alpha \ni x_m$, and put $\sigma_m(n) = \alpha$; otherwise, pick $C_\beta \in \mathcal{C}_n$ with $C_\beta \ni x_m$, and put $\sigma_m(n) = \beta$. Then, $f(\sigma_m) = x_m$. Let $K = \{\sigma_m \mid m \in \mathbb{N}\} \cup f^{-1}(x)$. Then, $f(K) = L \cup \{x\}$. We show K is compact in the metric space M. Since $f^{-1}(x) = \Pi B_n$ is compact in M, it suffices to show that any subsequence $\{\sigma_{m(i)} \mid i \in \mathbb{N}\}$ of $\{\sigma_m \mid m \in \mathbb{N}\}$ has an accumulation point in $f^{-1}(x)$. Let $S = \{x_{m(i)} \mid i \in \mathbb{N}\}$. Since $\mathcal{C}_1$ satisfies (c_1), for each subsequence T of S, some $C_T \in \mathcal{C}_1$ is contains x and a point of T. Thus, $St(x, \mathcal{C}_1)$ contains S eventually (or frequently). But, $\mathcal{C}_1$ is a point-finite. Then, there exist $\sigma(1) \in B_1$, and a subsequence N_1 of $\{m(i) \mid i \in \mathbb{N}\}$ such that, for all $k \in N_1$, $\sigma_k(1) = \sigma(1)$. For $n \in \mathbb{N}$, similarly, there exist $\sigma(n) \in B_n$, and a subsequence N_n of N_{n-1} such that, for all $k \in N_n$, $\sigma_k(n) = \sigma(n)$. Then, the sequence $\{\sigma_{m(i)} \mid i \in \mathbb{N}\}$ accumulates to the point $\sigma = (\sigma(n)) \in f^{-1}(x)$. Then, the map f is pseudo-sequence-covering. Thus, f is also quotient by Fact 4. $\square$

We recall that a space X is *symmetric* (respectively *semi-metric*) if there exists a non-negative real valued function d defined on $X \times X$ such that $d(x, y) = 0$ iff $x = y$; $d(x, y) = d(y, x)$ for all $x, y \in X$; and $G \subset X$ is open in X iff, for each $x \in G$, $S_{1/n}(x) \subset G$ (respectively $x \in \operatorname{int} S_{1/n}(x) \subset G$) for some $n \in \mathbb{N}$, where $S_{1/n}(x) = \{y \in X \mid d(x, y) < 1/n\}$. Semi-metric spaces are precisely Fréchet, symmetric spaces. Symmetric spaces are g-first countable, thus, sequential by Fact 1. Every quotient π-image of a metric space is symmetric [2].

Remark 5. (1) Michael and Nagami [22] asked whether every quotient s-image Y of a metric space must be a compact-covering, quotient s-image of some metric space. Recently, H. Chen [5] gave a negative answer: There exists a (Hausdorff) space Y such that Y is a quotient two-to-one image of a locally separable metric space, but Y is not any compact-covering, quotient s-image of any metric space. While, in [11], it was shown that every quotient s-image of a metric space must at least be a “pseudo-sequence-covering” quotient s-image of some metric space. Theorem 4 shows that the same situation also arises among quotient compact images of metric spaces.

(2) A sequence $\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ ($\mathcal{C}_{n+1} \prec \mathcal{C}_n$) of covers of a space X is called a semi-refined sequence if, for any $x \in X$, $\{St(x, \mathcal{C}_n) \mid n \in \mathbb{N}\}$ is a weak nbd base at x. It is shown that a space X is a quotient compact image of a metric space iff X has a semi-refined sequence of point-finite covers ([13] or [17]). Thus, using Proposition 3 and Lemma 1, the equivalence among (b)–(e) in Theorem 4 are also shown.

(3) In the following, (i) shows that the conditions of the covers $\mathcal{C}_n$ or the $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ are essential, and (ii) and (iii) are related to (e) in Theorem 4.

(i) Every space has a σ-strong cs-network. Also, not every quotient finite-to-one (hence, compact) image of a locally compact metric space has a point-countable cs-network or a point-countable weak base; see [21, Remark 14(2)].

(ii) It is impossible to replace "strong cs^*-network" by "cs^*-network", or "(strong) cs-network". Indeed, every first countable space need not be a quotient, compact (or π-) image of a metric space even if it has a base $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ such that each $\mathcal{C}_n$ is a point-finite cover with $\mathcal{C}_{n+1} \prec \mathcal{C}_n$, and for any $n \in \mathbb{N}$, $\bigcup\{\mathcal{C}_m \mid m \geqslant n\}$ is a base (hence, cs-network). To show this, let X be a Lindelöf space with a σ-disjoint base, but X has a closed subset which is not a G_δ-set (see Example 6.4 in [6], for example). Then, X has the obvious base $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ satisfying the above conditions (here, $\mathcal{C}_n \ni X$, and $\mathcal{C}_n \supset \bigcup\{\mathcal{C}_m \mid m < n\}$). While, every first countable symmetric space is semi-metric, thus, every closed subset is a G_δ-set. Then, X is not symmetric. Thus, X is not any quotient, compact (or π-) image of a metric space. Also, see the latter of (i).

(iii) It is impossible to omit "$\mathcal{C}_{n+1} \prec \mathcal{C}_n$ for each $n \in \mathbb{N}$" (also, in the definition of the σ-strong networks) in view of the Lindelöf space X in (ii).

Concerning the characterizations for quotient compact images of locally compact (or locally separable) metric spaces, we have Theorem 6 below. A characterization for quotient, compact (or s-) images of locally separable metric spaces is also obtained in [20]. However, we don't know whether every quotient compact image of a locally separable metric space must be a pseudo-sequence-covering, quotient compact image of a locally separable metric space.

Let X be a space, and S be a sequence converging to $x \in X$. Let us say that a cover $\mathcal{C}$ of X is a *cs^*-network for the sequence S*, if for any nbd U of x, and any subsequence T of S, there exists $C \in \mathcal{C}$ such that $x \in C \subset U$, and C contains a subsequence T' of T.

Theorem 6.

(1) *For a space X, the following* (a)–(c) *are equivalent.*

(a) *X is a compact-covering, quotient compact image of a locally compact metric space.*

(b) *X is a quotient compact image of a locally compact metric space.*

(c) *X is determined by a point-finite cover $\mathcal{C}$ of compact metric subsets.*

(2) *For a space X, the following* (a)–(c) *are equivalent.*

(a) *X is a quotient compact image of a locally separable metric space.*

(b) *X is a sequential space with a point-finite cover $\{X_\alpha \mid \alpha \in A\}$, where each X_α has a σ-point-finite strong network $\bigcup\{\mathcal{C}_{\alpha_n} \mid n \in \mathbb{N}\}$ of countable covers satisfying the following*:

($*$) *For each sequence S in X converging to $x \in X$, there exist an $\alpha \in A$, and a subsequence T of S such that any $St(x, \mathcal{C}_{\alpha_n})$ contains T eventually.*

(c) *Same as in (b), but replace "any $St(x, \mathcal{C}_{\alpha_n})$ contains T eventually" by "$\bigcup\{\mathcal{C}_{\alpha_n} \mid n \in \mathbb{N}\}$ is a cs^*-network for T" in ($*$).*

Proof. (1) (a) $\Rightarrow$ (b) is clear. (b) $\Rightarrow$ (c) holds by Fact 3. For (c) $\Rightarrow$ (a), let $f : S \to X$ be the quotient map defined by the cover $\mathcal{C}$ (see Fact 2(1)). Then, f is quotient compact such that X is locally compact metric. Thus, f is compact-covering by [23, Proposition 4.8]. Hence, (a) holds.

(2) For (a) $\Rightarrow$ (b), let $f : M \to X$ be a quotient compact map such that M is locally separable metric. Then, as is well known, M is the topological sum of separable metric spaces $M_\alpha (\alpha \in A)$. Since f is compact, $\{X_\alpha \mid \alpha \in A\}$ is a point-finite cover of X. Note that each f is a quotient map such that each $f|M_\alpha$ is a compact map with M_α separable metric. Then, it is routine to see that each $X_\alpha = f(M_\alpha)$ has the obvious σ-point-finite strong network $\bigcup\{\mathcal{C}_{\alpha_n} \mid n \in \mathbb{N}\}$ of countable covers, and $\bigcup\{\mathcal{C}_{\alpha_n} \mid n \in \mathbb{N}\}$ satisfies ($*$) by Fact 4. For (b) $\Rightarrow$ (a), for each $\alpha \in A$, let $M_\alpha = \{(\sigma(n)) \in \Pi I_{\alpha_n} \mid \{C_{\sigma(n)} \in \mathcal{C}_{\alpha_n} \mid n \in \mathbb{N}\}$ is a network for some point $x_\sigma \in X_\alpha\}$, here $\mathcal{C}_{\alpha_n} = \{C_\gamma \mid \gamma \in I_{\alpha_n}\}$. Then, each M_α is locally separable metric. Let M be the topological sum of separable metric spaces M_α ($\alpha \in A$). Then, M is locally separable metric. Let $f : M \to X$ be the obvious map. Then, f is compact. While, by ($*$), f is subsequence-covering as in the proof of Theorem 4. Thus, f is quotient by Fact 4, for X is sequential. (b) $\Rightarrow$ (c) is obvious, and (c) $\Rightarrow$ (b) holds in view of (f) $\Rightarrow$ (e) $\Rightarrow$ (b) in Lemma 1. $\square$

Next, we give characterizations for *sequence-covering*, quotient compact images of metric spaces. We need the following definition. Referring to [1] (or [7]), we say that a family $\mathcal{C}$ of subsets of a space X is point-regular ($=$ *PF*-regular of [14]), if for any $x \in X$, and any open nbd U of x, $\{C \in \mathcal{C} \mid C \ni x$ and $C \cap (X - U) \neq \emptyset\}$ is at most finite; equivalently, for any closed subset F of X, $\{C \in \mathcal{C} \mid C \cap F \neq \emptyset\}$ is point-finite at any point of $X - F$.

For a cover $\mathcal{C}$ of a space X, we use the following notations: $\mathcal{C}_x = \{C \in \mathcal{C} \mid C \ni x\}$, and $\mathcal{C}^m = \{C \in \mathcal{C} \mid$ if $C' \in \mathcal{C}$ with $C \subset C'$, then $C' = C\}$.

Lemma 7. *Let $\mathcal{C}$ be a point-regular cover of a space X. Then the following hold.*

(1) *$\mathcal{C}_x$ is point-finite at any point $X - \{x\}$.*
(2) *For each $C \in \mathcal{C}$, some $A \in \mathcal{C}^m$ contains C.*
(3) *If $\mathcal{M} \subset \mathcal{C}_x$ is infinite, then $\mathcal{M}$ is a network at x.*
(4) *Let $\mathcal{C}$ be a cs^*-network for X.*

Then $\mathcal{C}$ is point-countable, and $\mathcal{C}$ can be expressed as $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where $\mathcal{C}$ is a σ-strong network, and any $\mathcal{C}_n$ satisfies (c_2).

Proof. (1) is clear. For (2), for some $C_1 \in \mathcal{C}$, suppose not. Then there exists an infinite sequence $\{C_n \mid n \in \mathbb{N}\} \subset \mathcal{C}$ such that $C_n \subset C_{n+1}$, but $C_n \neq C_{n+1}$. Thus, all C_n contain points x and y in X with $x \neq y$, a contradiction to (1).

For (3), if $\mathcal{M}$ is not a network at x, then, for some open nbd U of x, there exists distinct elements $C_n \in \mathcal{M}$ $(n \in \mathbb{N})$ with $C_n - U \neq \emptyset$. Pick $x_n \in C_n - U$. Let $F\,\mathrm{cl}(\{x_n \mid n \in \mathbb{N}\})$. Then, $x \notin F$, but $\{C \in \mathcal{C} \mid C \cap F \neq \emptyset\}$ is not point-finite at x, a contradiction.

For (4), first we show $\mathcal{C}$ is point-countable. Suppose some $\mathcal{C}_x$ is uncountable. Let $\mathcal{C}_x = \{B_\alpha \mid \alpha \in A\}$, and pick $y_\alpha \in B_\alpha - \{x\}$ for each $\alpha \in A$. Since $\mathcal{C}_x$ is uncountable, by (1), there exist $k \in \mathbb{N}$, and a subsequence L of $\{y_\alpha \mid \alpha \in A\}$ such that $\mathrm{ord}(q, \mathcal{C}_x) = k$ for all points $q \in L$. By (3), L converges to x. Since $\mathcal{C}$ is a cs^*-network, some $C_1 \in \mathcal{C}_x$ contains a subsequence L_1 of L. Pick $p_1 \in L_1$. Since L_1 converges to $x \in X - \{p_1\}$, some $C_2 \in \mathcal{C}_x$ contains a subsequence L_2 of L_1, but C_2 does not contain p_1. Pick $p_2 \in L_2$. Since L_2 converges to $x \in X - \{p_1, p_2\}$, some $C_3 \in \mathcal{C}_x$ contains a subsequence L_3 of L_2, but, C_3 does not contain p_1 nor p_2. In this way, at $k+1$ steps, some $C_{k+1} \in \mathcal{C}_x$ contains a subsequence L_{k+1} of L_k. Then, the sets C_n $(n \leqslant k+1)$ are distinct, thus, for $q \in L_{k+1} \subset L$, $\mathrm{ord}(q, x) \geqslant k+1$, a contradiction. Thus, $\mathcal{C}$ is point-countable. Next, we show that there exist covers $\mathcal{C}_n$ of X $(n \in \mathbb{N})$ such that $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, $\mathcal{C}$ is a σ-strong network for X, and any $\mathcal{C}_n$ satisfies (c_2). We can assume that $\mathcal{C}$ is closed under finite intersections. Let $\mathcal{C}_1 = \mathcal{C}^m$. Then, $\mathcal{C}_1$ is a cover of X by (2). Let $\mathcal{C}' = (\mathcal{C} - \mathcal{C}^m) \cup \{\{x\} \mid x \in X\}$. Then, $\mathcal{C}'$ is a point-regular cs^*-network for X. Indeed, we only show $\mathcal{C}'$ is a cs^*-network for X. Let L be a sequence converging to $x \in L$, and let U be an open nbd of x. Since $\mathcal{C}$ is a cs^*-network, by the same way as in the above, there exist $C_1, C_2, C \in \mathcal{C}$ such that $x \in C = C_1 \cap C_2 \subset C_1 \cup C_2 \subset U$, and C contains a subsequence of L. But, $C \neq C_1$, hence $C \in \mathcal{C}'$. Thus, $\mathcal{C}'$ is a cs^*-network for X. Let $\mathcal{C}_2 = [\mathcal{C}']^m$. Then, $\mathcal{C}_2$ is a cover of X by (2). Let $\mathcal{C}'' = (\mathcal{C} - (\mathcal{C}_1 \cup \mathcal{C}_2)) \cup \{\{x\} \mid x \in X\}$. Thus, $\mathcal{C}'' = \{\mathcal{C}' - \mathcal{C}'^m\} \cup \{\{x\} \mid x \in X\}$, and $\mathcal{C}'$ is a point-regular cs^*-network. Then, $\mathcal{C}''$ is a point-regular cs^*-network as is seen above. Let $\mathcal{C}_3 = [\mathcal{C}'']^m$. In this way, we can get covers $\mathcal{C}_n$ $(n \in \mathbb{N})$ of X with $\mathcal{C}_{n+1} \prec \mathcal{C}_n$. For every $C \in \mathcal{C}$, by (1), only finitely many sets $C' \in \mathcal{C}$ satisfy $C \subset C'$ with $C \neq C'$, so that $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$. And, by (3), $\mathcal{C}$ is a σ-strong network. Besides, all $\mathcal{C}, \mathcal{C}', \mathcal{C}'', \ldots$ are cs^*-networks for X, hence, these satisfy (c_2). Then, all $\mathcal{C}_n$ satisfy (c_2) by (2). Thus, (4) holds. □

Lemma 8. *Let X be a g-first countable space. Let $\mathcal{C}$ be a point-countable cs-network for X which is closed under finite intersections. Then, X has a weak base which is a subcover of $\mathcal{C}$; see the proof of Lemma* 7(3) *in* [21].

Theorem 9. *The following are equivalent for a space X.*

(a) *X has a point-regular weak base.*
(b) *X is a g-first countable with a point-regular cs-network.*
(c) *X has a σ-point-finite strong weak base.*
(d) *X is a sequential space with a σ-point-finite strong cs-network.*
(e) *X is a sequential space with a σ-point-finite strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_3).*
(f) *X is a sequence-covering, quotient compact-image of a metric space.*

Proof. (a) $\Rightarrow$ (b): This holds by means of Lemma 7(4) and Fact 1.

(b) $\Rightarrow$ (a): This holds by Lemma 8.

(a) $\Rightarrow$ (c): Let $\mathcal{C}$ be a point-regular weak base for X, and let $\mathcal{C}_1 = \mathcal{C}^m$. Since $\mathcal{C}$ is point-regular, $\mathcal{C}_1$ is a cover of X by Lemma 7(2). To see $\mathcal{C}_1$ is point-finite, suppose not at $x \in X$. Since $\mathcal{C}$ is a cs^*-network by Fact 1, we can put $(\mathcal{C}^m)_x = \{C_n \mid n \in \mathbb{N}\}$ by Lemma 7(4). Let $C \in \mathcal{C}$ be a weak nbd of x. Then, by Lemma 7(2), some C_n, for example, C_1 contains C. Since $C_{n+1} \not\subset C_1$, pick $x_n \in C_{n+1} - C_1$. Then, some subsequence L of $\{x_n \mid n \in \mathbb{N}\}$ converges to x by Lemma 7(3). But, by Fact 1, L is eventually in C, then so is in C_1, a contradiction. Hence $\mathcal{C}_1$ is a point-finite cover of X. Let $\mathcal{C}' = (\mathcal{C} - \mathcal{C}^m) \cup \eta(X)$, where $\eta(X)$ is the set of all isolated points in X. Then $\mathcal{C}'$ is a point-regular weak base for X. Indeed, we only show that $\mathcal{C}'$ is a weak base for X. Let $x \in X$ be not isolated in X, and let U be an open nbd of x. Then, for some weak nbd $V \in \mathcal{C}$ of x, $V \subset U$ and V contains $y \neq x$. Pick a weak nbd $W \in \mathcal{C}$ of x such that $W \subset V$, but $y \notin W$. Then, $W \in \mathcal{C}'$, and W is a weak nbd of x with $W \subset U$. Hence, $\mathcal{C}'$ is a weak base. For each $n \geqslant 2$, let $\mathcal{C}_n = [(\mathcal{C} - \bigcup\{\mathcal{C}_j \mid j = 1, \ldots, n-1\}) \cup \eta(X)]^m$. Then, each $\mathcal{C}_n$ is a point-finite cover in view of the above. And, $\mathcal{C} = \bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, and also, $\mathcal{C}$ is a σ-strong network by Lemma 7(3). Hence, $\mathcal{C}$ is a σ-point-finite strong weak base for X.

(c) $\Rightarrow$ (d): Note that X is g-first countable. Hence, (d) holds by Fact 1.

(d) $\Rightarrow$ (e): This holds by the same way as in Lemma 1(3).

(e) $\Rightarrow$ (f): Let $f : M \to X$ be the map defined by the σ-point-finite strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ (see Fact 2(2)). Then f is a map from the metric space M onto X. And, f is compact, for $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ is a σ-point-finite strong network for X. To see f is sequence-covering, let $L = \{x_n \mid n \in \mathbb{N}\}$ be a sequence converging to x. Since each $\mathcal{C}_n$ satisfies (c_3), there exists $C_{\sigma(n)} \in \mathcal{C}_n$ such that $C_{\sigma(n)}$ contains x, and L eventually. Let $\sigma = (\sigma(n))$. Then $f(\sigma) = x$. Let $x_m \in L$. For each $n \in \mathbb{N}$, if the $C_{\sigma(n)}$ contains x_m, let $\sigma_m(n) = \sigma(n)$, otherwise, take a $C_{\alpha(n)} \in \mathcal{C}_n$ with $C_{\alpha(n)} \ni x_m$, and let $\sigma_m(n) = \alpha(n)$. Let $\sigma_m = (\sigma_n(n))$ for each $m \in \mathbb{N}$. Then $f(\sigma_m) = x_m$, and the sequence $\{\sigma_n \mid n \in \mathbb{N}\}$ converges to the point σ. Hence, f is sequence-covering, and thus, f is quotient by Fact 4. (f) $\Rightarrow$ (b): Let $f : M \to X$ be a sequence-covering, quotient compact map with M metric. For each $n \in \mathbb{N}$, let $\mathcal{C}_n$ be a locally finite open refinement of $\{S_{1/n}(p) \mid p \in M\}$ such that $\mathcal{C}_{n+1} \prec \mathcal{C}_n$. Let $\mathcal{C} = \bigcup\{f(\mathcal{C}_n) \mid n \in \mathbb{N}\}$. Since f is sequence-covering, compact, it is routinely shown that $\mathcal{C}$ is a σ-point-finite cs-network for X. Since f is moreover quotient, for any $x \in X$, $\{St(x, \mathcal{C}_n) \mid n \in \mathbb{N}\}$ is a weak base at x. Thus, X is g-first countable. Also, for each open nbd U of $x \in X$, some $St(x, f(\mathcal{C}_n)) \subset U$, which implies that $\{C \in \mathcal{C} \mid C \ni x,\ C \cap (X - U) \neq \emptyset\}$ is finite since any $\mathcal{C}_n$ is point-finite. Hence, X is a g-first countable space with a point-regular cs-network $\mathcal{C}$. $\square$

Corollary 10. *The following are equivalent for a space X.*

(a) *X is an open, compact image of a metric space.*

(b) *X is a meta-compact, developable space.*

(c) *X is a first countable space with a σ-point-finite cs-network, and any closed subset of X is a G_δ-set.*

(d) *X is a Fréchet space with a σ-point-finite strong cs-network.*
(e) *X is a Fréchet space with a point-regular weak base.*

Proof. (a) $\Longleftrightarrow$ (b) is well-known (indeed, (a) $\Rightarrow$ (b) is routinely, (b) $\Rightarrow$ (a) is shown by means of Fact 2(2), for X has a development of point-finite covers). (b) $\Rightarrow$ (c) is routinely, and (d) $\Rightarrow$ (e) holds by Theorem 9. For (e) $\Rightarrow$ (b), let $\bigcup\{\mathcal{T}_x \mid x \in X\}$ be a point-regular weak base for X. Since X is Fréchet, for any $T \in \mathcal{T}_x$, $x \in \operatorname{int} T$ by Fact 1. Thus, X has a point-regular base. Thus, (b) holds by [7, Lemma 5.4.7], for example. For (c) $\Rightarrow$ (d), X is a first countable space with a σ-point-finite cs-network, then X has a σ-point-finite weak base by Lemma 8. Since X is first countable, X has a σ-point-finite base. While, any closed subset of X is a G_δ-set. Thus, X has a development of point-finite covers (see the proofs of Theorems 8.5 and 8.6 in [10]). Hence, (d) holds. □

Let us give characterizations for certain quotient π-images of metric spaces. Let (X, d) be a symmetric space. A sequence $\{x_n \mid n \in \mathbb{N}\}$ in X is called *d-Cauchy* if, for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geqslant k$. X is called *Cauchy* (respectively *weak Cauchy*) if every convergent sequence is d-Cauchy (respectively every convergent sequence has a d-Cauchy subsequence).

Lemma 11. *Let (X, d) be a symmetric space, and let $\mathcal{C}$ be the cover of all convergent, d-Cauchy sequences in X. Then, the map $f : M \to X$ defined by the cover $\mathcal{C}$ is a π-map with respect to some metric in M; see the proof of Lemma* 2.2 *in* [28].

Associated with the conditions (c_i) $(i = 1, 2, 3)$ for a space X, let us define the following conditions (s_i) for convenience. (s_i): X has a σ-strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_i).

We have the following characterizations by means of the conditions (s_i).

Theorem 12. *For a space X,* (1)–(3) *below hold.*
(1) *X is a symmetric space iff X is a sequential space satisfying (s_1).*
(2) *The following are equivalent.*
  (a) *X is an weak Cauchy symmetric space.*
  (b) *X is a sequential space satisfying (s_2).*
  (c) *X is a quotient π-image of a* (*locally compact*) *metric space.*
(3) *The following are equivalent.*
  (a) *X is a Cauchy symmetric space.*
  (b) *X is a sequential space satisfying (s_3).*
  (c) *X is a sequence-covering, quotient π-image of a* (*locally compact*) *metric space.*

Proof. For (1), recall that a space is symmetric iff it has a semi-refined sequence [29]. Thus, (1) holds by Lemma 1. In (2) (respectively (3)), (a) $\Longleftrightarrow$ (c) is also shown in [15] (respectively [28]), but let us give its proof for the readers. Since (3) is similarly shown as in (2), we show (2) holds. (a) $\Rightarrow$ (c) holds by means of Lemma 11 and Fact 4. For

(c) $\Rightarrow$ (b), let $f : M \to X$ be a quotient, π-map with M metric. Then X is sequential since f is quotient. For each $n \in \mathbb{N}$, let $\mathcal{C}_n$ be an open refinement of $\{S_{1/n}(p) \mid p \in M\}$, and $\mathcal{C}_{n+1} \prec \mathcal{C}_n$. Since f is a quotient π-map, $\bigcup\{f(\mathcal{C}_n) \mid n \in \mathbb{N}\}$ is a σ-strong network for X, and any $f(\mathcal{C}_n)$ determines X by Fact 3. Hence, (b) holds by Lemma 1. For (b) $\Rightarrow$ (a), let $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$ be a σ-strong network for X such that any $\mathcal{C}_n$ satisfies (c_2). For $x, y \in X$, let $d(x, y) = \inf\{1/n \mid y \in St(x, \mathcal{C}_n)$ for some $n \in \mathbb{N}\}$. Then, in view of Lemma 1, (X, d) is a weak Cauchy symmetric space. □

Remark 13. (1) Not every sequential space with a σ-strong cs-network is symmetric by means of Remark 2. Thus, for (2) (respectively (3)) of Theorem 12, the analogous result obtained from omitting the σ-point-finiteness of the σ-strong network in (e) of Theorem 4 (respectively (d) of Theorem 9) does not hold.

(2) In view of Theorem 12, the implication $(c_1) \Rightarrow (c_2)$ (respectively $(c_2) \Rightarrow (c_3)$) does not hold for X, even if X is a sequential space with a σ-strong network $\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_1) (respectively (c_2)). Indeed, not every symmetric space is weak Cauchy [15] (respectively not every weak Cauchy semi-metric space is Cauchy by (A) and (B) in the following (3)).

(3) Around Cauchy symmetric spaces, we recall the following results, where (D) holds by Combining Theorem 12 with (A) and (C). For the definition of g-developable spaces, see [16].

(A) A space is developable iff it is Cauchy semi-metric [3].
(B) Every semi-metric space is weak Cauchy semi-metric [4].
(C) A space is g-developable iff X is Cauchy symmetric [16].
(D) A space X is g-developable (respectively developable) iff X is a sequential (respectively Fréchet) space satisfying (s_3).

Theorem 14. *For a space X, the following hold.*

(1) *X is a quotient s- and π-image of a metric space iff X is a sequential space with a σ-point-countable strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ determines X.*
(2) *X is a sequence-covering, quotient s- and π-image of a metric space iff X is a sequential space with a σ-point-countable strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_3).*

Proof. (1) The "only if" part is shown as in the proof of the implication (c) $\Rightarrow$ (b) in Theorem 12, but the $\mathcal{C}_n$ there are assumed to be locally finite. For the "if" part, let $f : M \to X$ be the map defined by the σ-strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$. Then, f is an s-map since each $\mathcal{C}_n$ is point-countable. Here, a metric d in the 0-dimensional metric space M is defined as follows: for $\sigma_1, \sigma_2 \in M$, let $d(\sigma_1, \sigma_2) = 1/k$, where $k = \min\{n \in \mathbb{N} \mid \sigma_1(n) \neq \sigma_2(n)\}$. Then, the map f is a π-map with respect to the metric d. Indeed, for each open nbd U of x, there exists $n \in \mathbb{N}$ such that $St(x, \mathcal{C}_n) \subset U$. Then, $d(f^{-1}(x), M - f^{-1}(U)) > 0$. To see the map f is quotient, it suffices to show that f is subsequence-covering by Fact 4. Let $L = \{x_n \mid n \in \mathbb{N}\}$ be a sequence converging to $x \notin L$. Since each $\mathcal{C}_n$ determines X, by Lemma 1, there exists $C_{\sigma(n)} \in \mathcal{C}_n$ such that $C_{\sigma(n)}$ contains x, and a subsequence L_n of

L_{n-1}, here $L_0 = L$. Let $\sigma = (\sigma(n)) \in M$. For each $m \in \mathbb{N}$, define $\sigma_m = (\sigma_m(n)) \in M$ as follows: if $C_{\sigma}(n) \ni x_m$, put $\sigma_m(n) = \sigma(n)$; otherwise, pick $C_\beta \in \mathcal{C}_n$ with $C_\beta \ni x_m$, and put $\sigma_m(n) = \beta$. Then, $f(\sigma_m) = x_m$, and the sequence $\{\sigma_n \mid n \in \mathbb{N}\}$ accumulates to the point σ. Since M is metric, some subsequence K of $\{\sigma_n \mid n \in \mathbb{N}\}$ converges to σ. Then, $f(K)$ is a subsequence of L. Thus, f is subsequence-covering.

(2) This is similarly shown as in (1), but the "if" part holds in view of the proof of the implication (c) $\Rightarrow$ (d) in Theorem 9. □

Recall that a *closed* cover $\mathcal{C}$ of a space X is a *closed k-network* if, for any compact subset K and any open subset U with $K \subset U$, there exists a finite $\mathcal{C}' \subset \mathcal{C}$ such that $K \subset \bigcup \mathcal{C}' \subset U$. Every closed k-network is a cs^*-network. In the following corollary, (2) is shown in [24] when a Fréchet space X has a point-regular closed k-network.

Corollary 15.

(1) *Let X be a sequential space. If X has a point-regular cs^*-network (or point-regular closed k-network), then X is a quotient s- and π-image of a metric space.*
(2) *Let X be a Fréchet space. If X has a point-regular cs^*-network, then X is a developable space with a point-countable base.*

Proof. (1) holds by Theorem 14(1) and Lemmas 1 and 7(4). For (2), X is sequential. Thus, X is a quotient π-image of a metric space by (1). Then, X is symmetric. Thus, X is semi-metric, for X is Fréchet. While, X is a quotient s-image of a metric space by (1). Then, X has a point-countable base by [8]. Hence, X is a developable space with a point-countable base by [12]. □

We give some characterizations for certain images of metric spaces when these images are not necessarily sequential. For example, we have the following by reviewing the proofs in this section.

Proposition 16. *For a space X, the following* (1)–(3) *hold.*

(1) (a) *X is a pseudo-sequence-covering, compact (respectively s-) image of a locally compact metric space iff X has a point-finite (respectively point-countable) cover $\mathcal{C}$ of compact metric subsets, where $\mathcal{C}$ satisfies (c_2).*
  (b) *X is a sequence-covering, compact (respectively s-) image of a locally compact metric space iff X has a point-finite (respectively point-countable) cover of compact metric subsets, where $\mathcal{C}$ satisfies (c_3).*
(2) (a) *X is a pseudo-sequence-covering, compact image of a metric space iff X has a σ-point-finite strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_2).*
  (b) *X is a sequence-covering, compact image of a metric space iff X has a σ-point-finite strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_3).*
(3) (a) *X is a subsequence-covering, π-image of a metric space iff X has a σ-strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_2).*
  (b) *X is a sequence-covering, π-image of a metric space iff X has a σ-strong network $\bigcup\{\mathcal{C}_n \mid n \in \mathbb{N}\}$, where any $\mathcal{C}_n$ satisfies (c_3).*

(4) *In* (3)*, we can add a prefix* “s-” *before* “π-” *by adding* “*the* $\mathcal{C}_n$ *are moreover point-countable*”*. In particular, every space with a point-regular* cs^**-network is the subsequence,* s*- and* π*-image of a metric space.*

We will give examples illustrating the main theorems of this paper.

Example 17.

(1) A developable space Y satisfying the following properties.
 (a) (i) Y is a sequence-covering, compact-covering, and open compact image of a metric space.
  (ii) Y is also a sequence-covering, compact-covering, open compact image of a (0-dimensional) metric subspace M of a *Baire space*.
 (b) Y is a sequence-covering, compact-covering, quotient π-image of a locally compact metric space.
 (c) Y is not any quotient s-image of any locally separable metric space.

(2) A symmetric space Y satisfying the following properties.
 (a) (i) Y is a compact-covering, quotient compact image of a locally compact metric space.
  (ii) Y is also a quotient compact image of a (0-dimensional) metric subspace M of a *Baire space*.
 (b) Y is not any sequence-covering, quotient π-image of any metric space.
 (c) Y is not any sequence-covering, quotient s-image of any metric space.

Proof. (1) Let Y be the developable space Y in [26, Example 3.2]. Namely, for each real number r, let $X_r = \{(x, y) \mid y = |x - r|\}$ be a subset of the upper half plane, and define a base on X_r as follows: $\{(x, y)\}$ if $y > 0$; and $\{(x, y) \mid y = |x - r| < 1/n\}$, $n \in \mathbb{N}$. Let Y be a space determined by a cover $\mathcal{C} = \{X_r \mid r \text{ is real}\}$. Thus, $\mathcal{C}$ is a point-finite, closed and open cover of Y. Let X be the topological sum of $\mathcal{C}$. Then X is metric. Then, the obvious map from X onto Y is sequence-covering, compact-covering, open, and finite-to-one. Thus, (a)(i) holds. Thus, (a)(ii) holds in view of Corollary 10. Here, note that every open map defined on a metric space is sequence-covering. For (b), let T be the topological sum of all convergent sequences in Y. Then, T is locally compact metric. Since Y is developable, the obvious map f from T onto X is a sequence-covering, quotient π-map by Remark 13(3) and Lemma 11. Also, the map f is compact-covering. Indeed, let K be a compact subset of Y. Let $K_1 = \bigcup\{K \cap X_r \mid K \cap X_r \text{ is finite}\}$, and let $K_2 = \bigcup\{K \cap X_r \mid K \cap X_r \text{ is infinite}\}$. Then, K_1 is closed discrete in Y, thus K_1 is finite. We note that, for each real r, $K \cap X_r$ is a sequence converging to the point r if it is not finite, and also, $\{(r, 0) \mid r \text{ is real}\}$ is closed discrete in Y. Thus, K_2 is a finite union of convergent sequences. Hence, $K = K_1 \cup K_2$ is a finite union of convergent sequences. This implies that the map f is compact-covering. Hence, (b) holds. To show (c) holds, assume that Y is a quotient s-image of a locally separable metric space L. Since L is determined by a point-countable open cover of separable subsets, Y is determined by a point-countable cover of separable subsets by

Fact 4. Then, since Y is first countable, Y is locally separable by [11, Lemma 2.6]. But, Y is not locally separable. This is a contradiction. Thus, (c) holds.

(2) Let Y be the symmetric space Y in [21, Remark 14(2)] (or [28, Example 2.14(3)]). Namely, let X be the topological sum of $\{I, S_\alpha \mid \alpha \in I\}$, I is the closed unit interval, and each S_α is a convergent sequence. Then, X is locally compact metric. Let Y be the quotient space obtained from X by identifying the limit point of S_α with $\alpha \in I$ for each $\alpha \in I$. Then, the obvious map from X onto Y is quotient, and compact-covering since every compact subset of Y is contained in a finite union of I and S_α's. Then, (a)(i) holds. Thus, (a)(ii) holds in view of Theorem 4. Since X is not g-developable by [28, Example 2.14(3)], (b) holds by Theorem 12(3) and Remark 13(3). For (c), note that every sequence-covering, quotient s-image of a metric space has routinely a point-countable cs-network. But, Y has no point-countable cs-networks by [21, Remark 14(2)]. Thus, (c) holds. □

Finally, we pose questions below. (1) is affirmative if the $\mathcal{C}$ is a σ-strong network (see Theorem 4); or X is Fréchet (cf. Corollary 10). If we replace "symmetric space" by "first countable space", (1) is negative by (ii) in Remark 5(3).

Questions.

(1) *Let X be a symmetric space with a σ-point-finite cs-network $\mathcal{C}$. Is X a quotient, compact image of a metric space*?

(2) *For a sequential space X with a point-regular cs^*-network* (*or point-regular cs-network*)*, characterize X by means of a nice image of a metric space.*

Addendum

As for (2), recently S. Lin and P. Yan showed that the following (1) and (2) are equivalent for a space X. Thus, the following (1), (2), and (3) are equivalent by Theorem 9. (1) X is a sequential space with a point-regular cs-network. (2) X is a point-regular weak base. (3) X is a sequence-covering quotient compact image of a metric space.

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